On the Bound States in a Non-linear Quantum Field Theory of a Spinor Field with Higher Derivatives[1]

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11th July 1997

Abstract

We consider a model of quantum field theory with higher derivatives for a spinor field with $(\bar{\varphi}\varphi)^2$ selfinteraction. With the help of the Bethe–Salpeter equation we study the problem of the two particle bound states in the "chain" approximation. The existence of a scalar bound state is established.

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1 Introduction

One of the basic problems in Quantum Field Theory (QFT) is related to the presence of divergences in the S-matrix elements. The reason for their appearance is well known: the propagators do not tend sufficiently quick to zero at large momenta. In standard QFT where one deals with equations of motion for Fermi (Boze) fields which are of first (second) order in the derivatives, the only solution of this problem is given by the renormalization method. The core of this method consists in the following – due to the arbitrariness in the S-matrix [1], one is able to add to the Lagrangean counterterms, which compensate the divergences. But it is well known that this method is not universal – there exist nonrenormalizable theories (which contain infinite number of different types divergent Feynman diagrams). An example is the theory of the weak interactions, proposed by Fermi. Its nonrenormalizability has led after long investigations to the creation of the Weinberg-Sallam theory.

One possible way to overcome the problem with divergences in QFT is to reject the restriction about the order of derivatives in the Lagrangean (or in the equation of motion) i.e., to consider Higher Derivative QFT (HDQFT). The main advantage of HD theories is that propagators have better behaviour at large momenta. That means that a theory containing a certain number of divergences in the standard case, in the HD case become (partially or completely) free from them.

Higher derivatives (HD) are not something new in field theory. It is known since a long time that HD could be used for regularization; different reguralarizations [2] are equivalent to the usage of HD [3]. Regularization with higher derivatives is convenient to use in the analysis of Yang-Mills fields [4]. HD have been used in classical electrodynamics [5] for elimination of the problem with the infinite field–energy of the electron.

In the present paper we deal with a HDQFT for a spinor field with scalar selfinteraction. The equation of motion for this field is of third order in the derivatives. The coupling constant is dimensionless (with respect to the term with highest derivative), which is peculiar to the Yang-Mills models. An important role in the model plays an additional mass-parameter with large (but finite) value. This parameter guarantees that the action is dimensionless and spinor fields have the standard dimension i.e. $3/2$. As we shall see below, the propagator in our model coincides with a spinor propagator regularized by

Pauli-Villars, and the additional parameter plays the role of the regularization mass. That is why we suppose that this parameter is much larger than the mass of the physical field presented in the theory. After quantization, this parameter represents the mass of the spinor fields with nonpositive norm - the so called HD "ghosts" [6].

In the nonrelativistic limit the equation of motion coincides with the nonlinear Schrödinger equation. The basic hypothesis in our work is that (at least for low energies) the equation of motion has solutions of the Schrödinger type, i.e., two-particle bound state. The main goal of the present article is to verify this hypothesis. In order to do that, we use an analogue of the Bethe-Salpeter equation [7].

2 The model

In the present paper we discuss a model of QFT with third order (in the derivatives) Lagrangean $L \sim \bar{\varphi}(\Box - M^2)(\partial_\mu \gamma^\mu - m)\varphi$. However, for the derivation of the commutation relations and Green functions it is more convenient to work with the following (equivalent to the initial) second order Lagrangean (with an additional spinor field ψ):

$$\begin{aligned} L &= \frac{1}{\sqrt{M^2 - m^2}}(\partial^\mu \bar{\psi}\partial_\mu \varphi + \partial^\mu \bar{\varphi}\partial_\mu \psi) + \bar{\varphi}(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - m)\varphi \\ &\quad + \bar{\psi}(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu + m)\psi - \frac{m^2}{\sqrt{M^2 - m^2}}(\bar{\psi}\varphi + \bar{\varphi}\psi) + \frac{g}{M^2 - m^2}(\bar{\varphi}\varphi)^2 \end{aligned} \quad (1)$$

where $A\overleftrightarrow{\partial}_\mu B = A\partial_\mu B - \partial_\mu A B$. This Lagrangean contains an additional spinor field $\psi(x)$, and M is the arbitrary but finite mass parameter mentioned above. The coupling constant g is dimensionless and we shall assume it has arbitrary value and sign.

The equations of motion for both fields φ and ψ are:

$$\begin{aligned} \frac{(\Box + m^2)}{\sqrt{M^2 - m^2}}\psi &= (i\gamma^\mu \partial_\mu - m)\varphi + \frac{2g}{M^2 - m^2}(\bar{\varphi}\varphi)\varphi \\ \frac{(\Box + m^2)}{\sqrt{M^2 - m^2}}\varphi &= (i\gamma^\mu \partial_\mu + m)\psi \end{aligned} \quad (2)$$

After exclusion of the auxiliary field ψ from (2) we obtain a covariant equation of motion for the physical field only:

$$(\Box + M^2)(i\gamma^\mu \partial_\mu - m)\varphi + 2g(\bar{\varphi}\varphi)\varphi = 0 \quad (3)$$

Let us consider its nonrelativistic limit. For this, it is convenient to rewrite (3) in the form:

$$\left(\frac{\Box + m^2}{M^2 - m^2} + 1\right)(i\gamma^\mu \partial_\mu - m)\varphi(x) + \frac{2g}{M^2 - m^2}(\bar{\varphi}\varphi)\varphi(x) = 0$$

In momentum representation the first term has the form:

$$\left(\frac{-p^2 + m^2}{M^2 - m^2} + 1\right)(\hat{p} - m)\varphi(p)$$

Because $M^2 >> m^2$, we see, that for low momenta the term $\frac{-p^2+m^2}{M^2-m^2}$ is extremely small. Consequently, for low energies the behaviour of (3) is determined by the equation:

$$(i\gamma^\mu \partial_\mu - m)\varphi + \frac{2g}{M^2 - m^2}(\bar{\varphi}\varphi)\varphi = 0 \tag{4}$$

This is a non-linear Dirac equation. Following the standard procedure, it is easy to find its nonrelativistic limit:

$$\left(i\frac{\partial}{\partial t} + \frac{\Delta^2}{2m}\right)\varphi' + \frac{2g}{M^2 - m^2}(\bar{\varphi}'\varphi')\varphi' = 0 \tag{5}$$

Here φ' is the large component of the spinor field φ with mass m. The equation (5) is equivalent to the two-particle Schrödinger equation with contact potential. The latter has the form:

$$i\frac{\partial}{\partial t}\Psi(x_1, x_2; t) = \mathcal{H}(x_1 - x_2)\Psi(x_1, x_2; t) \tag{6}$$

where:

$$\mathcal{H} = -\frac{1}{2m}\sum_{i=1}^{2}\nabla_i^2 + \lambda\delta(x_1 - x_2) = \mathcal{H}_0 + \mathcal{H}_{int}$$

Equation (6) has a unique solution when the coupling constant λ has negative sign [9]. This suggests, that at least in the domain of low energies, the equation (3) could describe bound states of two relativistic particles.

3 Quantization of the model

The quantization of the model is made in the interaction representation. This procedure consists in the replacement of the field functions with operators. We assume all operators are normal ordered, and for simplicity we omit the sign for normal ordering. The free equation of motion for the field φ is:

$$\frac{(\Box + M^2)}{M^2 - m^2}(i\gamma^\mu \partial_\mu - m)\varphi = 0 \tag{7}$$

The anticomutator $\Gamma_{\alpha\beta}(x) = \{\varphi_\alpha(x), \bar{\varphi}_\beta(0)\}$, obeys the following initial conditions (see Appendix A):

$$\Gamma_{\alpha\beta}(x)|_{x_0=0} = 0 \; ; \; \partial_0 \Gamma_{\alpha\beta}(x)|_{x_0=0} = 0 \; ; \; \partial_0^2 \Gamma_{\alpha\beta}(x)|_{x_0=0} = (M^2 - m^2)\gamma^0_{\alpha\beta}\delta^3(x) \tag{8}$$

It is obvious that $\Gamma(x)$ satisfy (7). The solution of (7) with initial conditions (8) is:

$$\Gamma_{\alpha\beta}(x) = -i(i\gamma^\mu \partial_\mu + m)_{\alpha\beta}(D_m(x) - D_M(x)) \tag{9}$$

where $D_m(x)$ is the Pauli-Jordan function for mass m.

The propagator of the field φ, here denoted by $\mathcal{S}^c_{\alpha\beta}(x) = \langle T(\varphi_\alpha(x)\bar{\varphi}_\beta(0))\rangle_0$, is the solution of the equation:

$$\frac{(\Box + M^2)}{M^2 - m^2}(i\gamma^\mu \partial_\mu - m)i\mathcal{S}^c(x) = -\delta(x) \tag{10}$$

In momentum representation the solution of (10) is: $(k^\mu\gamma_\mu + m)\left(\frac{1}{m^2-k^2} - \frac{1}{M^2-k^2}\right)$ and the propagator is:

$$\mathcal{S}^c_{\alpha\beta}(x) = -i(i\gamma^\mu \partial_\mu + m)_{\alpha\beta}(D^c_m(x) - D^c_M(x)) \tag{11}$$

where:

$$D^c_m(x) = \frac{1}{(2\pi)^4}\int \frac{e^{-ikx}}{m^2 - k^2 - i\varepsilon} d^4k$$

is the causal Green function of the scalar field. The advantage of the usage of HD is obvious from the form (11) of the $\mathcal{S}^c$: the propagators in our model coincide with the ones in the standard theory of a spinor field regularized by Pauli-Villars. Now, it is easy to convince ourselves that in the present theory there are no divergences connected to the Feynman diagrams.

The solution of (7) could be written as:

$$\begin{aligned}\varphi_\alpha(x) &= \varphi^m_\alpha(x) \\ &+\frac{1}{(2\pi)^{3/2}}\int d^3k\{e^{ikx}(N_+u^{r,+}_\alpha(k)a^+_r(k)+N_-v^{r,+}_\alpha(k)c^+_r(k)) \\ &+e^{-ikx}(N_+u^{r,-}_\alpha(k)a^-_r(k)+N_-v^{r,-}_\alpha(k)c^-_r(k))\}\end{aligned} \tag{12}$$

where $\varphi^m(x)$ is the usual spinor field satisfying equation: $(i\gamma^\mu\partial_\mu - m)\varphi^m(x) = 0$. The mode-vectors $u(k)$ and $v(k)$ obey the equations:

$$\begin{aligned}(\mp k^\mu\gamma_\mu - M)u^{r,\pm}(k) &= 0 \\ (\mp k^\mu\gamma_\mu + M)v^{r,\pm}(k) &= 0\end{aligned}$$

and the usual normalising conditions

$$\bar{u}^{r,\pm}(k)u^{s,\mp}(k) = \pm\frac{M}{\Omega}\delta^{r,s} = \bar{v}^{r,\mp}(k)v^{s,\pm}(k)$$

$$\begin{aligned}\sum_s u^{s,\pm}_\alpha(k)\bar{u}^{s,\mp}_\beta(k) &= \frac{(k^\mu\gamma_\mu \mp M)_{\alpha\beta}}{2\Omega} \\ \sum_r v^{r,\pm}_\alpha(k)\bar{v}^{r,\mp}_\beta(k) &= \frac{(k^\mu\gamma_\mu \pm M)_{\alpha\beta}}{2\Omega}\end{aligned} \tag{13}$$

where $\Omega = \sqrt{\vec{k}^2 + M^2}$ and $N_\pm = \sqrt{\frac{M\pm m}{2M}}$. The creation and annihilation operators $a^\pm$ and $c^\pm$ in (12), obey the following anticommutation relations:

$$\begin{aligned}\{a^-_r(k), \overset{*}{a}{}^+_s(p)\} &= \{\overset{*}{a}{}^-_r(k), a^+_s(p)\} = -\delta_{rs}\delta(k-p) \\ \{c^-_r(k), \overset{*}{c}{}^+_s(p)\} &= \{\overset{*}{c}{}^-_r(k), c^+_s(p)\} = -\delta_{rs}\delta(k-p)\end{aligned} \tag{14}$$

According to eq.(14) these operators represent "ghosts" (with mass M) in the model. The physical subspace is given by vectors $\mid ph\rangle$, which satisfy the following additional conditions:

$$\begin{aligned}a^- \mid ph\Big\rangle &= \overset{*}{a}{}^- \mid ph\rangle = 0 \\ c^- \mid ph\Big\rangle &= \overset{*}{c}{}^- \mid ph\rangle = 0,\end{aligned}$$

4 The bound state equation

In 1951 Bethe and Salpeter derived an relativistic integral equation for the two-particle bound states [7]. With the help of this equation, some relativistic corrections to the fine and the superfine structure of the light atoms have been calculated [7], [8]. The kernel of this integral equation consists of all two-particle irreducible Feynman diagrams, but in the case of a small coupling constant [7] one can consider the kernel restricted to its first term (the so called "ladder" approximation). Applying their approach, we obtain an analogue equation for the interaction, considered in the paper. Here, this approximation leads to extreme simplification – due to the presence of a δ-functions, the integral equation is reduced to a set of 16 linear algebraic equations. In this approximation, the kernel corresponds to contact interaction of two particles, and could be written as:

$$K(x_1, x_2, x_3, x_4) = -\frac{ig}{M^2 - m^2}\delta(x_1 - x_3)\delta(x_2 - x_4)\delta(x_1 - x_2) \tag{15}$$

Hence, the analogue of the Bethe-Salpeter equation is:

$$\begin{aligned}\chi_{\alpha_1\alpha_2}(p; x_1, x_2) &= \int {G^0}_{\alpha_1\alpha_2\beta_1\beta_2}(x_1, x_2, x_3, x_4) K(x_3, x_4, x_5, x_6) \times \\ &\quad \chi_{\beta_1\beta_2}(p; x_5, x_6) dx_3 dx_4 dx_5 dx_6 \end{aligned} \tag{16}$$

where $\chi(p; x_1, x_2)$ is the relativistic two-particle wave function, defined as:

$$\chi_{\alpha\beta}(p; x_1, x_2) = \langle 0 \mid T(\varphi_\alpha(x_1)\varphi_\beta(x_2)) \mid p\rangle \tag{17}$$

Here p is the 4-momenta of the system, and

$${G^0}_{\alpha_1\alpha_2\beta_1\beta_2}(x_1, x_2, x_3, x_4) = \mathcal{S}^c_{\alpha_1\beta_1}(x_1 - x_3)\mathcal{S}^c_{\alpha_2\beta_2}(x_2 - x_4)$$

is the product of two free Green functions (11) corresponding to the two particles. Using the transformation properties of the wave function:

$$\chi(p; x_1, x_2) = e^{ipa}\chi(p; x_1 + a, x_2 + a)$$

(where a is an arbitrary 4-vector) and introducing absolute and relative coordinates X and x resp.:

$$\begin{aligned} X &= \frac{x_1 + x_2}{2} \\ x &= x_1 - x_2 \end{aligned}$$

we obtain: $\chi(p;x_1,x_2) = e^{-ipX}\chi(x)$. In the new co-ordinates, instead of eq.(16) we have:

$$\chi_{\alpha_1\alpha_2}(x) = \frac{ig}{(2\pi)^4(M^2-m^2)}\int d^4k\mathcal{S}^c_{\alpha_1\beta_1}(k)\mathcal{S}^c_{\alpha_2\beta_2}(p-k)e^{-i(k-p/2)x}\chi_{\beta_1\beta_2}(x=0) \tag{18}$$

One can show, that in the nonrelativistic limit eq. (18) coincides with the Schrödinger equation with δ-function potential. Indeed, for low energies (see (4)) in the momentum representation one can rewrite (18) as:

$$\left(\frac{\hat{p}}{2}+\hat{q}-m\right)_{\alpha_1\beta_1}\left(\frac{\hat{p}}{2}-\hat{q}-m\right)_{\alpha_2\beta_2}\tilde{\chi}_{\beta_1\beta_2}(q) = -\frac{ig}{M^2-m^2}\chi_{\alpha_1\alpha_2}(x=0)$$

where q is the relative momentum. Acting on both sides of the above equation with the projectors $\Lambda^+(\vec{q})\Lambda^-(\vec{q})$, where $\Lambda^\pm(\vec{q}) = \frac{1}{2}+\frac{(m\gamma^0\pm\vec{q}\gamma^0\vec{\gamma})}{2E(\vec{q})}$ we obtain, after simple transformations:

$$\left(\frac{E}{2}-E(\vec{q})+\varepsilon-i\delta\right)\left(\frac{E}{2}-E(\vec{q})-\varepsilon-i\delta\right)\tilde{\chi}^{+,-}(\varepsilon,\vec{q}) = -\frac{i\lambda}{M^2-m^2}\chi^{+,-}(x=0)$$

Here E is the energy in the rest frame ($\vec{p}=0$), ε is the relative energy, $E(\vec{q}) = \sqrt{(\vec{q})^2+m^2}$ and $\tilde{\chi}^{+,-}(\varepsilon,\vec{q}) = \Lambda^+(\vec{q})\Lambda^-(\vec{q})\tilde{\chi}(\varepsilon,\vec{q})$ (resp. for the function $\chi(0)$). The latter equation could be integrated with respect to ε. The result is:

$$(E-2E(\vec{q}))\int\tilde{\chi}^{+,-}(\varepsilon,\vec{q})\,d\varepsilon = \frac{g}{M^2-m^2}\chi^{+,-}(x=0)$$

In the nonrelativistic limit the projectors could be removed from the above (scalar) equation and we obtain exactly the Schrödinger equation in the momentum representation:

$$(E-2m)\tilde{\chi}(\vec{q}) = \frac{q^2}{m}\tilde{\chi}(\vec{q})+\frac{g}{M^2-m^2}\chi(x=0)$$

Let us return to the equation (18). It is an algebraic system connecting the wave function, taken in different points (x and 0). Consequently, if we set $x=0$ we obtain the following set of algebraic equations:

$$\chi_{\alpha_1\alpha_2}(x=0) = \mathcal{F}_{\alpha_1\beta_1\alpha_2\beta_2}(p)\chi_{\beta_1\beta_2}(x=0) \tag{19}$$

where:

$$\begin{aligned}\mathcal{F}_{\alpha_1\beta_1\alpha_2\beta_2}(p) &= \frac{ig}{(2\pi)^4(M^2-m^2)}\int\mathcal{S}^c_{\alpha_1\beta_1}(k)\mathcal{S}^c_{\alpha_2\beta_2}(p-k)d^4k \\ \mathcal{S}^c(k) &= -i(k^\mu\gamma_\mu+m)\left(\frac{1}{m^2-k^2-i\varepsilon}-\frac{1}{M^2-k^2-i\varepsilon}\right)\end{aligned} \tag{20}$$

The calculation of the integral (20) is long, but it is not connected to any principle difficulties. This integral is similar to the vacuum polarisation diagram in the Quantum Electrodynamics. Some details about the calculation of such integrals one can see in [1].

The final result can be written in the following form:

$$\mathcal{F} = a_{\mu\nu}\gamma^{\mu} \otimes \gamma^{\nu} + b_{\mu} \left(1\!\!1 \otimes \gamma^{\mu} + \gamma^{\mu} \otimes 1\!\!1 \right) + d \; 1\!\!1 \otimes 1\!\!1 \tag{21}$$

where the functions $a_{\mu\nu}, b_{\mu}$ and d in the rest frame $p = (E, \vec{0})$ are given in Appendix B. We are interested only in bound states between particles with mass m. This means, that the values of the energy must be in the interval $0 < E^2 < 4m^2$. It is easy to check, that in this interval $a_{\mu\nu}, b_{\mu}, d$ are analytic functions of the energy (the latter is not obvious only for the point $E^2 = 0$).

If one writes the wave function χ as column with 16 components, then the condition for compatibility of the set of algebraic equations(19)

$$det(\mathcal{F}_{16\times16} - 1\!\!1_{16\times16}) = 0 \tag{22}$$

determines whether there exist bound states or not.

5 Decomposition into Lorentz invariants

For the subsequent analysis it is instructive to decompose the two–particle wave function into Lorentz invariant quantities. First, we start with the observation, (see eq.(17)) that the quantity $\tilde{\chi}(x) = \chi(x) \left(C^T \right)^{-1}$, where C is the charge conjugation matrix, transforms as a tensor under bispinorial representation $\tau(\frac{1}{2}, 0) \oplus \tau(0, \frac{1}{2})$ of the Lorentz group, i.e.:

$$\tilde{\chi}'(x') = S^{-1}\tilde{\chi}(x)S \quad , \quad x'_{\mu} = \Lambda^{\nu}_{\mu} x_{\nu} \tag{23}$$

This representation is reducible, and it decomposes into a direct sum of irreducible representations: a scalar (s), a pseudoscalar (p), a 4-vector (V_{α}), an axial vector (A_{α}) and an asymmetric 2-rank tensor ($\Sigma_{\alpha\beta}$). Using the connection $S^{-1}\gamma^{\mu}S = \Lambda^{\mu}_{\nu}\gamma^{\nu}$, $\tilde{\chi}(x)$ could be written as:

$$\tilde{\chi} = s 1\!\!1 + p\gamma^5 + V_{\alpha}\gamma^{\alpha} + A_{\alpha}\gamma^5\gamma^{\alpha} + \Sigma_{\alpha\beta}\sigma^{\alpha\beta} \tag{24}$$

where $\sigma^{\alpha\beta} = \frac{i}{4}(\gamma^{\alpha}\gamma^{\beta} - \gamma^{\beta}\gamma^{\alpha})$. After simple transformations, from (21) and (19), one can obtain the equation for $\tilde{\chi}$:

$$a_{\mu\nu}\gamma^{\mu}\tilde{\chi}\gamma^{\nu} + b_{\mu} \left(\tilde{\chi}\gamma^{\mu} - \gamma^{\mu}\tilde{\chi} \right) + (1 - d)\tilde{\chi} = 0 \tag{25}$$

where we use that $C^{-1}\gamma_\mu C = -\gamma_\mu^T$. Finally, after standard γ-matrix algebra, from (25) we obtain the following equations for the Lorentz invariants:

$$
\begin{aligned}
(a_\mu^\mu - d + 1)s &= 0 \\
(a_\mu^\mu + d - 1)p &= 2b_\mu A^\mu \\
2a_\mu^\sigma V^\mu - (a_\mu^\mu + d - 1)V^\sigma &= 2b_\mu i\Sigma^{\mu\sigma} \\
2a_\mu^\sigma A^\mu - (a_\mu^\mu - d + 1)A^\sigma &= 2b^\sigma p \\
(a_\mu^\mu - d + 1)i\Sigma^{\sigma\rho} - 2(a^{\sigma\alpha} i\Sigma_\alpha^\rho - a^{\rho\alpha} i\Sigma_\alpha^\sigma) &= 2(b^\sigma V^\rho - b^\rho V^\sigma)
\end{aligned}
\tag{26}
$$

Hence, we see that in terms of fields s, p, V, A, Σ the matrix $\mathcal{F}_{16\times16}$ could be written as:

$$
\mathcal{F}_{16\times16} = \mathcal{F}_{1\times1}(s) \oplus \mathcal{F}_{5\times5}(p, A) \oplus \mathcal{F}_{10\times10}(V, \Sigma)
$$

Therefore, the condition for existing of bound states (22) decomposes to tree independent conditions:

$$
\mathcal{F}(s) - 1 = 0 \ \ , \ \ \det(\mathcal{F}(p, A) - 1\!\!1) = 0 \ \ , \ \ det(\mathcal{F}(V, \Sigma) - 1\!\!1) = 0 \tag{27}
$$

6 Study of the scalar bound state

Let us concentrate our attention on the conditions (27). It is clear that the condition for the scalar bound state is the simplest one, because the scalar field does not mix with the other fields. In this section we solve the condition for the scalar bound state, which (following (26)) is:

$$
a_\mu^\mu - d + 1 = 0 \tag{28}
$$

This equation is too complicated to be solved in the general case. But since we have a small parameter - E^2/M^2 - we can simplify eq.(28). For this purpose, we use the approximation of the functions $a_{\mu\nu}$ and d given into Appendix B. Then eq.(28) is:

$$
\begin{aligned}
1 \ + \ & \frac{g}{16\pi^2}\frac{M^2}{M^2 - m^2} + \frac{g}{16\pi^2}\frac{m^2}{M^2 - m^2}\left\{\left(\frac{E^2}{2m^2} - 3\right) ln\left(\frac{M^2}{m^2}\right)\right. \\
& \left. -1 + \left(4 - \frac{E^2}{m^2}\right)\frac{R_m}{E} arctg\left(\frac{E}{R_m}\right)\right\} + O\left(\frac{E^4}{M^4}\right) = 0
\end{aligned}
\tag{29}
$$

An interesting feature of this equation is that it contains two types of terms: the first two are of the order of 1, while the rest are $\sim m^2/M^2$ and therefore are much smaller than the first two. This means that (29) has solution if the large and small terms are equal to zero independently i.e. (29) decomposes in two different conditions for the large and small parts. These two equations are:

$$\frac{g'}{16\pi^2} + 1 = 0 \tag{30}$$

$$\Delta g + \left(\frac{E^2}{2m^2} - 3\right) ln\left(\frac{M^2}{m^2}\right) - 1$$

$$+ \left(4 - \frac{E^2}{m^2}\right)\frac{R_m}{E} arctg\left(\frac{E}{R_m}\right) = 0 \tag{31}$$

Remark: The separation of the eq.(29) in two equations is possible, if the large part is determined up to term of order m^2/M^2. This arbitrary term contributes in the equation for the small parts. In order to do that we represent t he coupling constant as: $g = g'\left(1 + \Delta g\frac{m^2}{M^2}\right)$. From (30) we have: $g' = -16\pi^2$. It is easy to verify that (31) has a unique solution which depends on M^2 and Δg. Indeed, the LHS of (31) is a monotonously increasing function of E, and at the ends of the interval $0 \leq E^2 \leq 4m^2$ it has the values:

$$LHS(31) = \begin{cases} \Delta g + 3 - 3ln\left(\frac{M^2}{m^2}\right) & E^2 = 0 \\ \Delta g - 1 - ln\left(\frac{M^2}{m^2}\right) & E^2 = 4m^2 \end{cases}$$

Consequently, the model considered here admits the existence of the scalar bound state with mass (in the rest frame) $E^2(\Delta g, M^2)$, which is the solution of eq.(31). This bound state exists for each $M^2 : M^2 >> m^2$, when the coupling constant is negative and has the value:

$$g = -16\pi^2\left(1 + \Delta g\frac{m^2}{M^2}\right)$$

where: $ln\left(\frac{M^2}{m^2}\right) + 1 < \Delta g < 3ln\left(\frac{M^2}{m^2}\right) - 3$. This result is also in correspondence with the solution of the non-linear Schrödinger equation, discussed above.

In conclusion, we want to mark the possibility for solutions of (27), connected to the other fields. It is easy to verify from eq.(26), that in the rest frame, the matrices $\mathcal{F}(p, A)$ and $\mathcal{F}(V, \Sigma)$, has only symmetrical placed nondiagonal terms (besides the main diagonal) which are $\sim m^2/M^2$. Consequently,

their contribution to the determinant is $\sim m^4/M^4$ or smaller. Because of this, we can consider these nondiagonal elements in these matrices as zeros. Finally we obtain equations of the type (29), for the four fields in consideration. To solve these equations we proceed in analogy with (30) and (31). In order to satisfy the (corresponding) condition for the large parts, we have to choose the constant g' to be:

$$g'_p = 16\pi^2 \quad , \quad g'_V = 32\pi^2 \quad , \quad g'_A = -32\pi^2$$

The root, corresponding to Σ field is not achievable, because the large component in this case is zero. It is important to note, that equations for small parts, corresponding to temporal and spatial components of the vector and axial-vector fields are different.

Appendix A

First, we introduce two auxiliary functions $\Delta_{\alpha\beta}(x) = \{\psi_\alpha(x), \bar{\psi}_\beta(0)\}$ and $S_{\alpha\beta}(x) = \{\varphi_\alpha(x), \bar{\psi}_\beta(0)\}$.

From the requirement that fields, separated with space-like interval are causal independent, it follows, that:

$$\Gamma_{\alpha\beta}(x)|_{x^0=0} = \Delta_{\alpha\beta}(x)|_{x^0=0} = S_{\alpha\beta}(x)|_{x^0=0} = 0 \tag{32}$$

The procedure of canonical quantization gives:

$$\{\pi_{\varphi_\alpha}(x), \varphi_\beta(0)\}|_{x^0=0} = \{\pi_{\psi_\alpha}(x), \psi_\beta(0)\}|_{x^0=0} = i\delta^3(x)\delta_{\alpha\beta} \tag{33}$$

where π is canonical conjugated momentum for corresponding field. Substituting in (33) the explicit form of conjugated momenta, we have:

$$\partial_0\Gamma_{\alpha\beta}(x)|_{x^0=0} = \partial_0\Delta_{\alpha\beta}(x)|_{x^0=0} = 0 \tag{34}$$

and

$$\partial_0 S_{\alpha\beta}(x)|_{x^0=0} = -i\sqrt{M^2 - m^2}\delta^3(x)\delta_{\alpha\beta} \tag{35}$$

Let us consider eq.(2) in the free case. One can show, that:

$$\frac{1}{\sqrt{M^2 - m^2}}\partial_0^2\Gamma_{\alpha\beta}(x)|_{x^0=0} = i\gamma^0_{\alpha\rho}\partial_0\{\psi_\rho(x), \bar{\varphi}_\beta(0)\}|_{x^0=0}$$

Finally, from the latter equation and from eq.(35) we obtain:

$$\partial_0^2\Gamma_{\alpha\beta}(x)|_{x^0=0} = (M^2 - m^2)\gamma^0_{\alpha\beta}\delta^3(x) \tag{36}$$

Appendix B

In the rest frame $p = (E, \vec{0})$, functions $a_{\mu\nu}, b_\mu$ and d in (21) are:

$$
\begin{aligned}
a_{\mu\nu} &= \frac{g}{16\pi^2 E^2 (M^2 - m^2)} \left\{ \frac{g_{\mu\nu}}{2} \left[-\frac{(M^2 - m^2)^2}{3} \right.\right. \\
&+ (3E^4 - 3E^2(M^2 + m^2) + (M^2 - m^2)^2) \frac{(M^2 - m^2)}{6E^2} ln\left(\frac{M^2}{m^2}\right) \\
&- \frac{1}{3E^2} R^3 \ arth\left(\frac{R}{M^2 + m^2 - E^2}\right) \\
&+ \frac{E}{3} R_m^3 arctg\left(\frac{E}{R_m}\right) \quad + \quad \left. \frac{E}{3} R_M^3 arctg\left(\frac{E}{R_M}\right) \right] \\
&+ g_{\mu 0} g_{\nu 0} \left[\frac{2}{3} (M^2 - m^2)^2 \right. \\
&+ (3E^2(M^2 + m^2) - 2(M^2 - m^2)^2) \frac{(M^2 - m^2)}{6E^2} ln\left(\frac{M^2}{m^2}\right) \\
&- \frac{1}{3E^2} (E^4 + E^2(M^2 + m^2) - 2(M^2 - m^2)^2) R \ \ arth\left(\frac{R}{M^2 + m^2 - E^2}\right) \\
&\left.\left. - \frac{E}{3} (2m^2 + E^2) R_m arctg\left(\frac{E}{R_m}\right) \quad - \quad \frac{E}{3} (2M^2 + E^2) R_M arctg\left(\frac{E}{R_M}\right) \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
d &= \frac{2m^2 g}{16\pi^2 E^2 (M^2 - m^2)} \left\{ \frac{(M^2 - m^2)}{2} ln\left(\frac{M^2}{m^2}\right) - R \ \ arth\left(\frac{R}{M^2 + m^2 - E^2}\right) \right. \\
&\left. - E R_m arctg\left(\frac{E}{R_m}\right) - E R_M arctg\left(\frac{E}{R_M}\right) \right\} \\
b_\mu &= \frac{E}{2m} g_{\mu 0} d
\end{aligned}
$$

where:

$$
R = \sqrt{(E^2 - M^2 - m^2)^2 - 4M^2 m^2} \quad R_m = \sqrt{4m^2 - E^2} \quad R_M = \sqrt{4M^2 - E^2}
$$

If we recall that E^2/M^2 is an extremely small parameter, then the above functions could be rewritten in the form:

$$
a_{\mu\nu} = \frac{g}{16\pi^2 (M^2 - m^2)} \left\{ g_{\mu\nu} \left[\frac{M^2}{4} + \frac{m^2}{2} \left[-ln\left(\frac{M^2}{m^2}\right) - \frac{5}{6} \right.\right.\right.
$$

$$
\begin{aligned}
& + \left(\frac{1}{6} ln \left(\frac{M^2}{m^2}\right) - \frac{1}{18}\right) \frac{E^2}{m^2} + \frac{1}{3}\left(4 - \frac{E^2}{m^2}\right) \frac{R_m}{E} arctg\left(\frac{E}{R_m}\right)\Bigg]\Bigg] \\
& + g_{\mu 0} g_{\nu 0} m^2 \left[\frac{2}{3} + \left(\frac{1}{6} ln\left(\frac{M^2}{m^2}\right) + \frac{1}{9}\right)\frac{E^2}{m^2} - \frac{1}{3}\left(2 + \frac{E^2}{m^2}\right)\frac{R_m}{E} arctg\left(\frac{E}{R_m}\right)\right]\Bigg\} \\
& + O\left(\frac{E^4}{M^4}\right) \\
d \ = \ & \frac{g}{16\pi^2(M^2 - m^2)}\, m^2 \left\{ ln\left(\frac{M^2}{m^2}\right) - 2\frac{R_m}{E} arctg\left(\frac{E}{R_m}\right)\right\} + O\left(\frac{E^4}{M^4}\right)
\end{aligned}
$$

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